ON A CLASS OF NON-SELF-ADJOINT PERIODIC BOUNDARY VALUE PROBLEMS WITH DISCRETE REAL SPECTRUM

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1. Introduction

We study the operator L_{per} defined by

$$L_{\mathrm{per}}u := i\epsilon(f(x)u'(x))' + iu'(x) \tag{1.1}$$

in which f is a given 2π-periodic function having the following properties:

$$f(x+\pi) = -f(x)\,, \qquad f(-x) = -f(x)\,; \tag{1.2}$$

and also

$$f(x) > 0 \qquad \text{for } x \in (0,\pi)\,. \tag{1.3}$$

In particular it follows that $f(\pi\mathbb{Z}) = 0$. We assume that f is continuous, and differentiable except possibly at a finite number of points, the points of non-differentiability excluding $\pi\mathbb{Z}$. We assume that $f'(0) = 2/\pi$ and that $0 < \epsilon < \pi$.

We consider (1.1) on the domain

$$\mathcal{D} = \{u \in L^2(-\pi,\pi) \,|\, L_{\mathrm{per}}u \in L^2(-\pi,\pi);\;\; u(-\pi) = u(\pi)\}. \tag{1.4}$$

Remark 1. Of course it is not obvious that functions $u \in L^2(-\pi,\pi)$ such that $L_{\mathrm{per}}u \in L^2(-\pi,\pi)$ have boundary values $u(\pm\pi)$; this was proved in [BLM], where we showed that if $u \in L^2(-\pi,\pi)$ and $L_{\mathrm{per}}u \in L^2(-\pi,\pi)$ then $u \in H^1(-\pi,\pi)$.

The main results of this paper are

Theorem 2. *The spectrum of operator L_{per} is*

(a) *real*
(b) *purely discrete, i.e. it consists only of isolated eigenvalues of finite multiplicity with no accumulation points apart from, possibly, infinity.*

Part (a) has been partially proved in [BLM], where we showed that all the *eigenvalues* (if they exist) are real. The rest of Theorem 2 follows from

Theorem 3. *The resolvent $(L_{\mathrm{per}} - \lambda)^{-1}$ is a compact operator on $L^2(-\pi,\pi)$ if λ is not an eigenvalue of L_{per}.*

Remark 4. The spectrum is always non-empty, as zero is an eigenvalue corresponding to a constant eigenfunction.

Date: March 2010.
The research of Michael Levitin was partially supported by the EPSRC grant EP/D054621.

In order to prove Theorem 3, we show that when λ is not an eigenvalue of L_{per} (which is guaranteed, for instance, if λ is not real) then the boundary value problem

$$i\epsilon(f(x)u'(x))' + iu'(x) - \lambda u(x) = F(x) \qquad -\pi < x < \pi \tag{1.5}$$

with periodic boundary conditions $u(-\pi) = u(\pi)$ has a unique solution $u \in \mathcal{D}$ for every $F \in L^2(-\pi, \pi)$.

The compactness of the resolvent is demonstrated by an "explicit" construction of a bounded Green function $G(x, s)$ such that $u(x) = \int_{-\pi}^{\pi} G(x, s)F(s)\,\mathrm{d}s$. The properties of G are established by studying the solutions of an associated homogeneous equation in Sections 2 and 3.

Motivation and scope of the present paper. Our interest in the operator (1.1) with domain determined by condition (1.2) is primarily motivated by [BeO'BSa] and [Da2], where $f(x) = (2/\pi)\sin x$, and therefore (1.5) takes the form

$$i\tilde{\epsilon}\left(\sin(x)u'(x)\right)' + iu'(x) - \lambda u(x) = F(x), \tag{1.6}$$

with $0 < \tilde{\epsilon} < 2$. This equation arises in fluid dynamics, and describes small oscillations of a thin layer of fluid inside a rotating cylinder. From a purely theoretical perspective, the eigenvalue problem associated to L_{per} has recently drawn a substantial amount of attention: see [ChPe], [We1], [We2], [DaWe] and [ChKaPy]. Despite the fact that (1.6) is highly non-self-adjoint, the spectrum of (1.6) consists exclusively of real eigenvalues of finite multiplicity, it is symmetric with respect to the origin and it accumulates at $\pm\infty$. It is also known that the eigenfunctions do not form an unconditional basis of $L^2(-\pi, \pi)$. Moreover, Davies and Weir [DaWe] have recently found explicit asymptotics of the eigenvalues as $\tilde{\epsilon} \to 0$.

In [BLM] we established that the eigenvalues of L_{per} are all real and form a symmetric set with respect to the origin. Theorem 2 above rules out completely the possibility of a non-empty essential spectrum and it answers an open question posed in our previous paper.

2. A naïve Frobenius analysis

Let p satisfy the integrating factor equation

$$\frac{p'}{p} = \frac{f'}{f} + \frac{1}{\epsilon f}. \tag{2.1}$$

Then $u(x)$ is a solution of (1.5) iff

$$(p(x)u'(x))' + \frac{i\lambda p}{\epsilon f}u(x) = -\left(\frac{ip}{\epsilon f}F\right)(x) \qquad -\pi < x < \pi. \tag{2.2}$$

For future use, we also recall the homogeneous differential equation

$$(p(x)u'(x))' + \frac{i\lambda p}{\epsilon f}u(x) = 0 \qquad -\pi < x < \pi \tag{2.2$'$}$$

In order to understand how p behaves near $x = 0$ and $x = \pi$ it is useful to consider a simple model. Suppose that near $x = 0$, the function f satisfies $f(x) = 2x/\pi$ – this maintains the normalization $f'(0) = 2/\pi$. Then (2.1) yields $\log p = \log(2x/\pi) + \pi/(2\epsilon)\log(x)$, whence $p(x) = Cx^{1+c/\epsilon}$ where C is an arbitrary non-zero constant and $c = \pi/2$. Similarly, near $x = \pi$, we can consider a simple model $f(x) =$

$(2/\pi)(\pi - x)$ and obtain $p(x) = \tilde{C}(\pi - x)^{1-c/\epsilon}$. In [BLM] we proved that, under the minimal assumptions (1.2) on f, the following results hold:
(2.3)

$$p(x) \sim \begin{cases} x^{1+c/\epsilon} & x \sim 0 \\ (\pi - x)^{1-c/\epsilon} & x \sim \pi, \end{cases} \qquad \text{and} \qquad \frac{p(x)}{f(x)} \sim \begin{cases} x^{c/\epsilon} & x \sim 0 \\ (\pi - x)^{-c/\epsilon} & x \sim \pi. \end{cases}$$

By considering the model $f(x) = 2x/\pi$, $p(x) = x^{1+c/\epsilon}$ in a neighbourhood of the origin and looking for solutions of the differential equation (2.2′) in the form $u(x) = x^{\nu}(1 + a_1 x + a_2 x^2 + \cdots)$ one can establish the asymptotic behaviour of solutions for this model in a neighbourhood of the origin; similarly near $x = \pi$. In [BLM] we show that, under hypotheses (1.2) on f, there exist solutions u of (2.2′) such that

$$u(x, \lambda) \sim \begin{cases} x^{-c/\epsilon} \text{ or } 1 & x \sim 0 \\ 1 \text{ or } (\pi - x)^{c/\epsilon} & x \sim \pi \\ 1 \text{ or } (x + \pi)^{c/\epsilon} & x \sim -\pi \end{cases}$$

This implies the existence of a unique solution $\phi(x) = \phi(x, \lambda)$ of (2.2′), such that

$$\phi(x, \lambda) \sim 1, \quad x \to 0. \tag{2.4}$$

The following result is crucial for reducing the problem to the interval $(0, \pi)$; in a sense it plays the same role in the analysis as the orthogonal splitting of the infinite matrix operator in Davies [Da2] in $\ell^2(\mathbb{Z})$ into three operators, hence reducing the problem to a problem in $\ell^2(\mathbb{N})$.

Lemma 5. *The solution ϕ has the symmetry property*

$$\phi(-x, \lambda) = \phi(x, -\lambda). \tag{2.5}$$

Proof. Define a function $v(x) = \phi(-x, \lambda)$. A direct calculation shows, thanks to the symmetry conditions (1.2), that v satisfies (2.2′) but with λ on the right hand side replaced by $-\lambda$. It also satisfies $v(0) = 1$. However eqn. (2.2′) has only one solution with this property, namely $\phi(x, -\lambda)$. This proves the result. □

We emphasize the important fact that λ is an eigenvalue of (2.2′) if and only if ϕ possesses the additional symmetry property

$$\phi(-\pi, \lambda) = \phi(\pi, \lambda). \tag{2.6}$$

In [BLM] we also show that there is also a second solution $\psi(x) = \psi(x, \lambda)$ of (2.2′) satisfying

$$\psi(x, \lambda) \sim \begin{cases} |x|^{-c/\epsilon} & x \sim 0 \\ |x \mp \pi|^{c/\epsilon} & x \sim \pm\pi \end{cases} \tag{2.7}$$

Observe that $\psi(-\pi, \lambda) = \psi(\pi, \lambda) = 0$, and that $\psi(x, \lambda)$ blows up when $x \sim 0$, at least when λ is not an eigenvalue.

Consequently, when λ is not an eigenvalue, we can also normalize $\psi(x, \lambda)$ by the condition

$$p(x)\psi'(x)\phi(x) - p(x)\phi'(x)\psi(x) = 1 \qquad -\pi < x < \pi.$$

The Wronskian in the right-hand side here is obviously a constant, and below we will always assume that $\psi(x, \lambda)$ satisfies this condition.

3. Construction of the Green function

Now, we are back to constructing explicitly the L^2 solution $u(x,\lambda)$ of (2.2) assuming that (2.6) fails. By the standard variation of parameters technique the general solution of (2.2) takes the form

$$
\begin{aligned}
u(x,\lambda) &= \psi(x,\lambda)\int_0^x \phi(s,\lambda)\left(\frac{-\mathrm{i}p}{\epsilon f}F\right)(s)\,\mathrm{d}s \\
&\quad + \phi(x,\lambda)\int_x^\pi \psi(s,\lambda)\left(\frac{-\mathrm{i}p}{\epsilon f}F\right)(s)\,\mathrm{d}s \\
&\quad + A\phi(x) + B\psi(x)\,,
\end{aligned}
\tag{3.1}
$$

where A and B are arbitrary complex constants.

It remains to check that one can choose constants A and B in such a way that

(a) $u \in L^2(-\pi,\pi)$,
(b) $L_{\mathrm{per}}u \in L^2(-\pi,\pi)$, and
(c) $u(-\pi,\lambda) = u(\pi,\lambda)$.

It is in fact sufficient, and easier, to check that one can choose constants A and B such that

(a') $u(x,\lambda)$ is continuous at $x=0$,
(b') $\mathrm{i}\epsilon f u' + \mathrm{i}u$ is continuous at $x=0$,

and (c) all hold. Indeed, (3.1) together with (a') implies (a), and together with (b') and (c) implies (b).

Note first, that by (a') and the behaviour of ψ near the origin, one is tempted to take $B=0$ in (3.1). We shall show that this choice is indeed the right one by the careful analysis of the remaining terms in (3.1).

By the Cauchy-Schwarz inequality and (2.3), (2.4), (2.7) we have,

$$
\begin{aligned}
&\left|\psi(x,\lambda)\int_0^x \phi(s,\lambda)\left(\frac{-\mathrm{i}p}{\epsilon f}F\right)(s)\,\mathrm{d}s\right| \\
&\le |\psi(x,\lambda)|\left(\int_0^x |\phi(s,\lambda)|^2\left|\frac{-\mathrm{i}p}{\epsilon f}\right|^2\mathrm{d}s\right)^{1/2}\left(\int_0^x |F(s)|^2\,\mathrm{d}s\right)^{1/2} \\
&\le C|x|^{-c/\epsilon}|x|^{c/\epsilon+1/2}\left(\int_0^x |F(s)|^2\,\mathrm{d}s\right)^{1/2} \\
&\le C|x|^{1/2}\|F\|\,.
\end{aligned}
\tag{3.2}
$$

Here, and throughout the rest of this paper, C denotes a generic positive constant; $\|\cdot\|$ is the standard norm in $L^2(-\pi,\pi)$.

Similarly,

$$
\begin{aligned}
&\left|\phi(x,\lambda)\int_x^\pi \psi(s,\lambda)\left(\frac{-\mathrm{i}p}{\epsilon f}F\right)(s)\,\mathrm{d}s\right| \\
&\le |\phi(x,\lambda)|\left(\int_x^\pi |\psi(s,\lambda)|^2\left|\frac{-\mathrm{i}p}{\epsilon f}\right|^2\mathrm{d}s\right)^{1/2}\left(\int_x^\pi |F(s)|^2\,\mathrm{d}s\right)^{1/2} \\
&\le C\cdot 1\cdot|\pi-x|^{1/2}\left(\int_x^\pi |F(s)|^2\,\mathrm{d}s\right)^{1/2} \\
&\le C|\pi-x|^{1/2}\|F\|\,,
\end{aligned}
\tag{3.3}
$$

so it is bounded. It is also continuous as the integrand is a product of a bounded function $\psi p/f$ and an L^2 function F.

Also, ϕ is continuous at zero, and ψ is not, and therefore (a') holds if and only if $B = 0$.

To check the condition (b'), it is now sufficient to verify that fu' is continuous at zero. Differentiating (3.1) with respect to x gives

$$\begin{aligned} fu'(x,\lambda) &= f\psi'(x,\lambda)\int_0^x \phi(s,\lambda)\left(\frac{-\mathrm{i}p}{\epsilon f}F\right)(s)\,\mathrm{d}s \\ &\quad + f\phi'(x,\lambda)\int_x^\pi \psi(s,\lambda)\left(\frac{-\mathrm{i}p}{\epsilon f}F\right)(s)\,\mathrm{d}s \\ &\quad + Af\phi'(x)\,, \end{aligned} \tag{3.4}$$

as the contributions from differentiating the integrals cancel out.

The last two terms go to zero as $x \to 0$, and we only need to check the continuity of the first term. Similarly to (3.2), we get

$$\begin{aligned} &\left| f(x)\psi'(x,\lambda)\int_0^x \phi(s,\lambda)\left(\frac{-\mathrm{i}p}{\epsilon f}F\right)(s)\,\mathrm{d}s\right| \\ &\le |f(x)\psi'(x,\lambda)| \left(\int_0^x |\phi(s,\lambda)|^2\left|\frac{-\mathrm{i}p}{\epsilon f}\right|^2\mathrm{d}s\right)^{1/2}\left(\int_0^x |F(s)|^2\,\mathrm{d}s\right)^{1/2} \\ &\le C|x||x|^{-c/\epsilon-1}|x|^{c/\epsilon+1/2}\left(\int_0^x |F(s)|^2\,\mathrm{d}s\right)^{1/2} \\ &\le C|x|^{1/2}\|F\|\,, \end{aligned} \tag{3.5}$$

which proves (b').

Finally, we need to guarantee that we can choose a value of constant A to ensure that condition (c) holds. Direct substitution, again taking account of (2.3), (2.4), (2.7) gives

$$\begin{aligned} u(\pi,\lambda) &= A\phi(\pi,\lambda)\,, \\ u(-\pi,\lambda) &= A\phi(-\pi,\lambda) + \phi(-\pi,\lambda)\int_{-\pi}^\pi \psi(s,\lambda)\left(\frac{-\mathrm{i}p}{\epsilon f}F\right)(s)\,\mathrm{d}s, \end{aligned}$$

and as λ is assumed not to be an eigenvalue, and therefore (2.6) is not satisfied, we can choose

$$A = \int_{-\pi}^\pi \psi(s,\lambda)\left(\frac{-\mathrm{i}p}{\epsilon f}F\right)(s)\,\mathrm{d}s \Big/ \left(\frac{\phi(\pi,\lambda)}{\phi(-\pi,\lambda)} - 1\right)\,. \tag{3.6}$$

This proves the existence of the resolvent of $(L_{\mathrm{per}}-\lambda)^{-1} : L_2(-\pi,\pi) \to L_2(-\pi,\pi)$ for each λ which is not an eigenvalue of L_{per}. Thus, the spectrum of L_{per} is pure point and real.

Now we proceed to writing down the expression for the Green function $G(x,s)$ of L_{per}. We note that (3.1) can be written, with account of $B = 0$ and (3.6), as

$$u(x,\lambda) = \int_{-\pi}^\pi G(x,s)F(s)\,\mathrm{d}s = \int_{-\pi}^\pi (G_{\mathrm{I}}(x,s) + G_{\mathrm{II}}(x,s) + G_{\mathrm{III}}(x,s))F(s)\,\mathrm{d}s$$

where we set $G(x,s) := G_{\mathrm{I}}(x,s) + G_{\mathrm{II}}(x,s) + G_{\mathrm{III}}(x,s)$, with

$$G_{\mathrm{I}}(x,s) := \begin{cases} \psi(x,\lambda)\phi(s,\lambda)\left(\frac{-\mathrm{i}p(s)}{\epsilon f(s)}\right) & \text{if } |x| \geq |s|\,, \\ 0 & \text{otherwise;} \end{cases}$$

$$G_{\mathrm{II}}(x,s) := \begin{cases} \phi(x,\lambda)\psi(s,\lambda)\left(\frac{-\mathrm{i}p(s)}{\epsilon f(s)}\right) & \text{if } x \leq s\,, \\ 0 & \text{otherwise;} \end{cases}$$

$$G_{\mathrm{III}}(x,s) := \phi(x,\lambda)\psi(s,\lambda)\left(\frac{-\mathrm{i}p(s)}{\epsilon f(s)}\right) \Big/ \left(\frac{\phi(\pi,\lambda)}{\phi(-\pi,\lambda)} - 1\right) \quad \text{for all } -\pi \leq x, s \leq \pi\,.$$

The functions $G_{\mathrm{II}}(x,s)$ and $G_{\mathrm{III}}(x,s)$ are bounded by (2.3), (2.4), (2.7), so we need to look at $G_{\mathrm{I}}(x,s)$. The only scope for trouble in the expression for G_{I} lies in the fact that $\psi(x,\lambda)$ blows up as $x \to 0$. However if x is small then, in the region $|x| \geq |s|$ where G_{I} is nonzero, (2.7) and (2.3) yield

$$\left|\psi(x,\lambda)\frac{p(s)}{f(s)}\right| \leq C|x|^{-c/\epsilon}|s|^{c/\epsilon} \leq C.$$

Thus G is bounded and hence is the kernel of a compact operator on $L^2(-\pi,\pi)$. Thus L_{per} has compact resolvent and purely discrete real spectrum.

4. Schatten class properties of the Green function

We first recall the standard notion of Schatten class operator. Let T be a compact operator and consider its (∞-dimensional) singular value decomposition (see e.g. [GoKr]):

$$T = \sum_{j=0}^{\infty} \alpha_j |v_j\rangle\langle w_j|$$

where the singular values $\alpha_j \geq 0$ and the two sets of vector $\{v_j\}$ and $\{w_j\}$ are orthonormal and not necessarily equal. Here we use the bra-ket notation: $|v\rangle\langle w|u = \langle u, w\rangle v$. For $p > 0$, we say that T is in the p-Schatten class, $T \in \mathcal{C}_p$, if

$$\|T\|_p := \left(\sum_{j=0}^{\infty} \alpha_j^p\right)^{1/p} < \infty.$$

Note that $\mathcal{C}_1$ are the trace class operators and $\mathcal{C}_2$ are the Hilbert-Schmidt operators.

Theorem 6. *The resolvent $(\lambda - L_{per})^{-1}$ is in $\mathcal{C}_p$ for all $p > 1$.*

Proof. Let $G_{\mathrm{I}}(x,s), G_{\mathrm{II}}(x,s), G_{\mathrm{III}}(x,s)$ be the components of the Green function $G(x,s)$. It suffices to show that each of the corresponding integral operators $R_{\mathrm{I}}(\lambda), R_{\mathrm{II}}(\lambda), R_{\mathrm{III}}(\lambda)$ is in $\mathcal{C}_p$.

We start by showing that $R_{\mathrm{III}}(\lambda) \in \mathcal{C}_1$. Indeed, $G_{\mathrm{III}}(x,s) = v(x)w(s)$ where

$$v(x) = \phi(x,\lambda) \quad \text{and} \quad w(s) = \psi(s,\lambda)\left(\frac{-\mathrm{i}p(s)}{\epsilon f(s)}\right) \Big/ \left(\frac{\phi(\pi,\lambda)}{\phi(-\pi,\lambda)} - 1\right).$$

By virtue of (2.4), (2.7) and (2.3), both $v, w \in L^2(-\pi,\pi)$. Then $R_{\mathrm{III}}(\lambda) = |v\rangle\langle w|$ and $\|R_{\mathrm{III}}(\lambda)\|_1 = \|v\|_{L^2(-\pi,\pi)}\|w\|_{L^2(-\pi,\pi)} < \infty$, as required.

Let us now show that $R_{\mathrm{II}}(\lambda) \in \mathcal{C}_p$ for all $p > 1$. Let $\Omega_{\mathrm{II}} = \{(x,s) \in [-\pi,\pi]^2 : x \le s\}$, be the supports of $G_{\mathrm{II}}(x,s)$. Decompose the characteristic function of Ω_{II} as

$$\mathbb{1}_{\Omega_{\mathrm{II}}}(x,s) = \sum_{j=0}^{\infty} \sum_{i=0}^{2^j-1} \mathbb{1}_{I_{2i,j}}(x)\mathbb{1}_{I_{2i+1,j}}(s)$$

where there are 2^{j+1} intervals

$$I_{i,j} = 2\pi \left[\frac{i}{2^{j+1}}, \frac{i+1}{2^{j+1}}\right] - \pi = \left[\frac{i\pi}{2^{j+1}} - \pi, \frac{(i+1)\pi}{2^{j+1}} - \pi\right], \quad i = 0, \dots, 2^{j+1}-1.$$

Then

$$G_{\mathrm{II}}(x,s) = \mathbb{1}_{\Omega_{\mathrm{II}}}(x,s)v(x)w(s) = \sum_{j=0}^{\infty} \sum_{i=0}^{2^j-1} v(x)\mathbb{1}_{I_{2i,j}}(x)w(s)\mathbb{1}_{I_{2i+1,j}}(s).$$

Let $S_{i,j} = |v\mathbb{1}_{I_{2i,j}}\rangle\langle w\mathbb{1}_{I_{2i+1,j}}|$. Then

$$\alpha_{i,j} := \|S_{i,j}\| = \|v\|_{L^2(I_{2i,j})}\|w\|_{L^2(I_{2i+1,j})} \le \frac{m}{2^j}$$

where m is independent of i and j. The constant m depends on λ and it is finite as a consequence of (2.4), (2.7) and (2.3). Let

$$S_j = \sum_{i=0}^{2^j-1} S_{i,j} = \sum_{i=0}^{2^j-1} \alpha_{i,j} \left| \frac{v\mathbb{1}_{I_{2i,j}}}{\|v\|_{L^2(I_{2i,j})}} \right\rangle \left\langle \frac{w\mathbb{1}_{I_{2i+1,j}}}{\|w\|_{L^2(I_{2i+1,j})}} \right|. \tag{4.1}$$

Since the intervals $I_{i,j}$ are pairwise disjoint for a fixed j, the right side of (4.1) is a singular value decomposition for S_j. Then

$$\|S_j\|_p = \left(\sum_{i=0}^{2^j-1} \alpha_i^p\right)^{1/p} \le \frac{m}{\left(2^{\frac{p-1}{p}}\right)^j}.$$

By the triangle inequality, this ensures that

$$\|R_{\mathrm{II}}(\lambda)\|_p < \infty$$

for all $p > 1$ as required.

The fact that $R_{\mathrm{I}}(\lambda) \in \mathcal{C}_p$ for all $p > 1$ follows by an analogous decomposition of the support $\Omega_{\mathrm{I}} = \{(x,s) \in [-\pi,\pi]^2 : |x| \ge |s|\}$ as the union of disjoint rectangles and a very similar argument.

□

Remark 7. Let λ_n be the eigenvalues of L_{per} and $\lambda \ne \lambda_n$. By virtue of [DuSc, cor.XI.9.7], the series $\sum_{n=0}^{\infty}(\lambda - \lambda_n)^{-p}$ converges absolutely and

$$\sum_{j=0}^{\infty} |\lambda - \lambda_n|^{-p} \le \|(\lambda - L_{\mathrm{per}})^{-1}\|_p^p \tag{4.2}$$

for all $p > 1$. According to the results of [We2] on the case $f(x) = (2/\pi)\sin x$, it is known that $\lambda_n \sim n^2$ as $n \to \infty$. Hence we know that $(\lambda - L_{\mathrm{per}})^{-1} \notin \mathcal{C}_{1/2}$. As $L_{\mathrm{per}} \ne L_{\mathrm{per}}^*$, the inequality in (4.2) can not generally be reverse for any $p > 0$. The question of whether $(\lambda - L_{\mathrm{per}})^{-1} \in \mathcal{C}_p$ for $p > 1/2$ will be addressed in subsequent work.

References

[BeO'BSa] E S Benilov, S B G O'Brien, and I A Sazonov, *A new type of instability: explosive disturbances in a liquid film inside a rotating horizontal cylinder*, J. Fluid Mech. **497**, 201–224 (2003).

[BLM] L Boulton, M Levitin and M Marletta, *A PT-symmetric periodic problem with boundary and interior singularities*, J. Diff. Eq., to appear (2010); preprint at http://arxiv.org/abs/0801.0172

[ChPe] M Chugunova and D Pelinovsky, *Spectrum of a non-self-adjoint operator associated with the periodic heat equation*, J. Math. Anal. Appl. **342**, 970–988 (2008).

[ChKaPy] M Chugunova, I Karabash and S Pyatkov, *On the nature of ill-posedness of the forward-backward heat equation*, Integral Equations and Operator Theory, to apper (2009); preprint http://www.arxiv2.org/articles/0803.2552/.

[Da1] E B Davies, *Spectral Theory and Differential Operators*, Cambridge University Press, Cambridge, 1995.

[Da2] E B Davies, *An indefinite convection-diffusion operator*, LMS J. Comp. Math. **10**, 288–306 (2007).

[DaWe] E B Davies and J Weir, *Convergence of eigenvalues for a highly non-self-adjoint differential operator*, Preprint http://arxiv.org/abs/0809.0787v1 (2008).

[DuSc] N Dunford and J Schwartz, *Linear Operators, part II: Spectral Theory*, Intertscience, New York (1957).

[GoKr] I Gohberg and M Krein, *Introduction to the theory of linear non-selfadjoint operators*, Translation of Mathematical Monographs 18, AMS, Providence (1969).

[We1] J Weir, *An indefinite convection-diffusion operator with real spectrum*, Appl. Math. Letters **22**, 280–283 (2009).

[We2] J Weir, *Correspondence of the eigenvalues of a non-self-adjoint operator to those of a self-adjoint operator*, preprint http://arxiv.org/abs/0801.4959v2 (2008).

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